The Higher Order Lifts on the Extended Vector Bundles

Cansel Yormaz¹, Ali Görgülü²

Department of Mathematics, Faculty of Science and Art, Pamukkale University, Denizli-Turkey¹
Department of Mathematics, Faculty of Science and Art, Osmangazi University, Eskişehir-Turkey²

Abstract: In this study, the higher order vertical and complete lifts of the tensor fields, which are defined on the extended vector bundles have been obtained.

Keywords: The extended vector bundles, The vertical and complete lifts of the tensor fields.

1. INTRODUCTION

In the previous studies [1-5], [11], [12] the first order lifts of the differential elements defined on a manifold M to its tangent bundle TM in the differential geometry have been obtained. Furthermore in the studies of [3] and [5], the second order lifts of the differential elements defined on a manifold M to its the second order tangent bundle TTM have been calculated. the structure of extended manifolds had been obtained. Especially, in the studies of [7], the second order lifts of the differential elements defined on a manifold M to its the second canonical extended manifold ²M are derived.

In §2 of this paper, we define the vertical and complete lifts of higher order of functions defined on the vector bundle π=(E,π,M) to its the extended vector bundles πk=(E,π,kM)[2].

We study in more detail the vertical and complete lifts of vector fields defined on the vector bundle π to its the extended vector bundles πk by considering the lifts of functions on the extended vector bundle πk in the §2.

In the §4, we calculate the vertical and complete lifts of 1-forms defined on the vector bundle π to its the extended vector bundles πk by considering the lifts of vector fields on the extended vector bundle πk in the §2.

In the §4, we compare the vertical and complete lifts of differential elements in the previous studies with the vertical and complete lifts of differential elements in this study. In this paper, all manifolds and mappings are assumed to be differentiable of class C∞, unless otherwise stated and the sum is taken over repeated indices.

2. THE Lifts OF FUNCTIONS

Let f:E → R be a function defined on the vector bundle π=(E,π,M). if we define the vertical lift of a function f defined on the vector bundle π to its the extended vector bundle πk=(E,π,kM) using the induction method for integer number k, then:

DEFINITION 2.1. Let f:E → R be a function defined on the vector bundle π then it is called the vertical lift of order k of f to the extended vector bundle Tπk-1 which a function fVk defined on the vector bundle Tπ k-1 defined with below the equation

\[ f^{V_k} = f \circ \tau_{0,E} \circ \tau_{1,E} \circ \ldots \circ \tau_{k-1,E} \]  

(2.1)

For all the tangent vector H∈T(π1E) and τ0,E,...,τk-1,E, therefore if we restrict the function \( f^{V_k} \) to kE, then the function \( f^{V_k} |_{kE} \) is defined on the extended vector bundle kE and it is called the vertical lift of order k of f to the extended vector bundle kE which is a function defined on the vector bundle Tπk-1 defined with below the equation

\[ f^{V_k} |_{kE} = f |_{\tau_{0,E}(\tau_{1,E}(...\tau_{k-1,E}(E))} \circ \tau_{0,E} \circ \tau_{1,E}(...\tau_{k-1,E}(E)) \]  

(2.2)
For shortness, we denote \( f^V_k \) with \( f^V_k|_{E_k} \). In that case; for all tangent vector \( H \in E \subset T^{(k-1)}E \) and 
\( \tau_0E \times \tau_{k-1}E \) \( (H) = e \in 0E; \)
\[ f^V_k(H) = f^V_k(H) = f(\tau_0E \circ \tau_{k-1}E(H)) = f(e). \]

**Proposition 2.1.** For all the functions \( f, g \in C_0(\alpha) \) and integer numbers \( 0 \leq k \leq \alpha \);
\[
\begin{align*}
\text{i)} \ (f + g)_r^V &= f_r^V + g_r^V \\
\text{ii)} \ (f \cdot g)_r^V &= f_r^V \cdot g_r^V
\end{align*}
\]
Furthermore, using the induction method, we define the complete lift of order \( k \) defined on its extended vector bundle \( \pi^k \). Let \( f \) be a function defined on the vector bundle \( \pi \). Then the differential of \( f \) is defined by
\[
\text{df} = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial u^\alpha} du^\alpha \quad (2.3)
\]
with respect to the adapted coordinates \( (x^i, u^\alpha) \) in \( E \).

Thus if we define the linear map
\[ t_1 : \mathcal{C}_1(\alpha^0) \rightarrow \mathcal{C}_1^0(T\mathcal{C}_1^0) \]
with respect to \( \text{Sp}(dx^i, du^\alpha : 1 \leq i \leq m, 1 \leq \alpha \leq n) = \mathcal{C}_1(\alpha^0) \) with \( t_1(dx^i) = x^i \), \( t_1(du^\alpha) = u^\alpha \).

**Definition 2.3.** Let \( f : E \rightarrow R \) be a function defined on the vector bundle \( \pi \) and the differential \( df \) of \( f \) be in form (2.3). Thus we write \( f^C \) for the function in the vector bundle \( T\pi^1 \) defined by
\[ f^C = t_1( df ) \quad (2.4) \]
and call \( f^C \) the complete lift of function \( f \) in \( \pi \) to vector bundle \( T\pi^1 \). The complete lift \( f^C \) has the local expression
\[ f^C = t_1( df ) = (\partial_{0i} f )^V x^i + (\partial_{0\alpha} f )^V u^\alpha \quad (2.5) \]
with respect to adapted coordinates \( (x^i, u^\alpha, x^i, u^\alpha) \) in \( T\pi, \) where \( \partial_{0i} f \) and \( \partial_{0\alpha} f \) denotes \( \frac{\partial f}{\partial x^i} \) and \( \frac{\partial f}{\partial u^\alpha} \) respectively. Therefore \( \pi^1 \) equal to \( T\pi \); if we restrict the complete lift \( f^C \) to the extended manifold \( E \); then we have
\[ f^C|_{-1} = f^C|_{TE} = (\partial_{0i} f )^V x^i + (\partial_{0\alpha} f )^V u^\alpha \quad (2.6) \]
therefore \( f^C|_{-1} \) is called the complete lift of function \( f \) in \( \pi \) to the extended vector bundle \( \pi^1 \).

In addition to for all tangent vector \( Z \in T^{(k-1)}E \) and \( \tau_0E(Z) = e \in 0E; \)
\[ f^C(1)^{(Z)} = (\partial_{0i} f )^V x^i + (\partial_{0\alpha} f )^V u^\alpha (2.7) \]
Thus we define the complete lift of order \( k \) of \( f \) in \( \pi \) to the vector bundle \( \pi^k \). Let \( \tilde{f} : k^{-1}E \rightarrow R \) be a function defined on the vector bundle \( \pi^{k-1} \). Then the differential of \( \tilde{f} \) is defined by
\[ d\tilde{f} = \frac{\partial \tilde{f}}{\partial x^i} dx^i + \frac{\partial \tilde{f}}{\partial u^\alpha} du^\alpha \quad (2.8) \]
with respect to the adapted coordinates \( (x^i, u^\alpha) \) \( 0 \leq \alpha \leq k \) \( \in \pi^{k-1} \).

Thus we define the linear map
\[ t_k : \mathcal{C}_1(\pi^k) \rightarrow \mathcal{C}_1^0(T\pi^{k-1}) \]
with respect to \( \text{Sp}(dx^i, du^\alpha : 0 \leq \alpha \leq k-1) = \mathcal{C}_1(\pi^k) \) with \( t_k(dx^i) = x^i \), \( t_k(du^\alpha) = u^\alpha \)

**Definition 2.4.** Let \( \tilde{f} \) be a function defined on the vector bundle \( \pi^{k-1} \) and the differential \( d\tilde{f} \) of \( \tilde{f} \) be in form (2.7). Thus we write \( \tilde{f}^C \) for the function in the vector bundle \( T\pi^{k-1} \) defined by
\[ \tilde{f}^C = t_k( d\tilde{f} ) \quad (2.9) \]
and call \( \tilde{f}^C \) the complete lift of function \( \tilde{f} \) in \( \pi \) to vector bundle \( T\pi^{k-1} \). The complete lift \( \tilde{f}^C \) has the local expression
\[ \tilde{f}^C = t_k( d\tilde{f} ) = (\partial_{\alpha i} \tilde{f} )^V x^i + (\partial_{\alpha\alpha} \tilde{f} )^V u^\alpha \quad (2.9) \]
with respect to the adapted coordinates \( (x^i, u^\alpha, x^i, u^\alpha) \) \( 0 \leq \alpha \leq k-1 \) \( \in T\pi^{k-1} \).
Let f be a function defined on the vector bundle \( \pi \) and the complete lift of order \( k \) \( f^{k-1}_k \) of f to the extended vector bundle \( \pi^k \). Then, therefore, the complete lift of order \( k \) \( f^{k-1}_k \) of f is defined on \( \pi^k \), if we take \( f \) equal to \( f^{k-1}_k \) and we write in (2.9) then we obtain

\[
( f^{k-1}_k )^C | k_E = f_k^C | r \alpha \mathbf{r} = (\partial^{\mathbf{r} \alpha}_f f^{k-1}_k )^C \mathbf{r} x^{r+\alpha} + (\partial^{\alpha} f^{k-1}_k )^C \mathbf{r} u^{r+\alpha}. \tag{2.10}
\]

Therefore, \( \pi^k \) is subvector bundle of \( \mathbb{T}\pi^k \) and \( k_E \) is submanifold of \( \mathbb{T}\pi(k-1)E \); we restrict the complete lift (\( f^{k-1}_k )^C \) to the total space \( k_E \) of \( \mathbb{T}\pi \) and we consider the adapted coordinates \((x^{\mathbf{r} \alpha}, \mathbf{r} u^{\alpha})=(x^{\mathbf{r} \alpha}, \mathbf{r} u^{\alpha})\) for \( 0 \leq \alpha \leq k-2 \) on the extended manifold \( k_E \) from the defining condition of \( k_E \), then we get

\[
( f^{k-1}_k )^C | k_E = f_k^C | r \alpha \mathbf{r} = (\partial^{\mathbf{r} \alpha}_f f^{k-1}_k )^C \mathbf{r} x^{r+\alpha} + (\partial^{\alpha} f^{k-1}_k )^C \mathbf{r} u^{r+\alpha}. \tag{2.11}
\]

For all tangent vector \( H \in k_E = \mathbb{T}k(k-1)E \) and \( \tau \)-k-1 \( \mathbb{T}E \) (\( \mathbb{T}\tau \)),

\[
f_k^{C'}(H) = f_k^{C'}(\tau \mathbb{T}E \tau \mathbb{T}k(k-1)(E)(H)) = f_{k^{C'}}(B). \]

**DEFINITION 2.5.** Let f be a function defined on the vector bundle \( \pi \). Then it is called the complete lift of order \( k \) \( f^k \) of the function f in \( \pi \) to its the extended vector bundle \( \pi^k \) that it is defined with equation (2.12).

**PROPOSITION 2.2.** For all the functions \( f, g \in \mathcal{A}_0 \) and integer numbers \( 0 \leq r \leq k \);

\[
\begin{align*}
&i) (f + g)_r^{C'} = f_r^{C'} + g_r^{C'} \\
&ii) (f \cdot g)_r^{C'} = \sum_{i=0}^{r} \binom{r}{i} f_r^{C''-i} g_r^{C''-i} \\
&iii) \left( \frac{\partial f}{\partial x_0^{\alpha}} \right)_r^{C'} = \frac{\partial f_r^{C'}}{\partial x_0^{\alpha}} = \frac{\partial f}{\partial x_0^{\alpha}} \\
&iv) \left( \frac{\partial f}{\partial u_0^{\alpha}} \right)_r^{C'} = \frac{\partial f_r^{C'}}{\partial u_0^{\alpha}}, \quad \left( \frac{\partial f}{\partial u_0^{\alpha}} \right)_r^{C''} = \frac{\partial f_r^{C''}}{\partial u_0^{\alpha}} f_r^{C'} \\
&v) \left( \frac{\partial f}{\partial x_0^{\alpha}} \right)_r^{C^{k-r}} = \frac{\partial f_k^{C}}{\partial x_0^{\alpha}} \quad \left( \frac{\partial f}{\partial u_0^{\alpha}} \right)_r^{C^{k-r}} - \frac{\partial f}{\partial u_0^{\alpha}} = \frac{\partial f_k^{C}}{\partial u_0^{\alpha}}. \\
\end{align*}
\]

**3. THE LIFTS OF VECTOR FIELDS**

Let X be a vector field defined on the vector bundle \( \square \) and local expression of X
\[ X = X^0_i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial u^\alpha}. \quad (3.1) \]

Thus if we define the vertical lift of a vector field \( X \) defined on the vector bundle \( \pi \) to its extended vector bundle \( \pi^k = (kE, \pi^k, M) \) using the induction method for integer number \( k \), then:

**DEFINITION 3.1.** Let \( X \) be a vector field defined on the vector bundle \( \square \). In that case; the vertical lift of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is a vector field defined on \( \pi^k \) which it is defined with the below equation

\[ X^V_k (t_{C_k}) = (X[f])_k^V; \quad \forall f \in C^0_{\square} \]. \( (3.2) \)

Now we give the below theorem connected with the components of the vertical lift of order \( k \) of \( X \).

**PROPOSITION 3.1.** Let \( X \) be a vector field defined on the vector bundle \( \square \) with form \( (3.1) \). In that case; the local expression of the vertical lift \( X^V_k \) of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^V_k = (X^o_i)_k^V \frac{\partial}{\partial x^i} + (X^\alpha_0)_k^V \frac{\partial}{\partial u^\alpha_0}. \quad \forall f \in C^0_{\square} \]. \( (3.3) \)

**PROOF:** Let \( X \) be a vector field defined on the vector bundle \( \square \) with form \( (3.1) \) and \( f \) be a function defined on the vector bundle \( \pi \). Then by considering Proposition 2.1. and 2.2. we write

\[ (X[f])_k^V = [X^0_i \frac{\partial f}{\partial x^i} + X^\alpha_0 \frac{\partial f}{\partial u^\alpha_0}]_k^V = (X^0_i)_k^V \]

\[ \frac{\partial f}{\partial x^i} + (X^\alpha_0)_k^V \frac{\partial}{\partial u^\alpha_0}. \quad (3.3) \]

If the local expression of the vertical lift \( X^V_k \) of \( k \) of \( X \) is

\[ X^V_k = X^{ri} \frac{\partial}{\partial x^{ri}} + X^{\alpha_0} \frac{\partial}{\partial u^{\alpha_0}}. \quad (3.4) \]

with respect to adapted coordinates \( \{ x^i, u^\alpha : 0 \leq r \leq k \} \) in \( kE \). Then we write

\[ X^V_k (t_{C_k}) = X^{ri} \frac{\partial f}{\partial x^{ri}} + X^{\alpha_0} \frac{\partial}{\partial u^{\alpha_0}}. \quad (3.5) \]

for all the function \( f \). Thus if we equalise the equation \( (3.3) \) with the equation \( (3.5) \) by considering the equation \( (3.2) \) then we obtain the below equations

\[ X^{ri} = 0, \quad X^{\alpha_0} = 0, \quad 0 \leq r \leq k-1, \quad X^{k_i} = (X^0_i)_k^V, \quad X^{\alpha_0} = (X^\alpha_0)_k^V. \]

Finally, the local expression of the vertical lift \( X^V_k \) of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^V_k = (X^0_i)_k^V \frac{\partial}{\partial x^i} + (X^\alpha_0)_k^V \frac{\partial}{\partial u^\alpha}. \quad (3.6) \]

Furthermore, we define the complete lift of order \( k \) of a vector field \( X \) defined on the vector bundle \( \pi \) to its extended vector bundle \( \pi^k = (kE, \pi^k, M) \) by using the induction method for integer number \( k \), then:

**DEFINITION 3.2.** Let \( X \) be a vector field defined on the vector bundle \( \square \). In that case; the complete lift of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is a vector field defined on \( \pi^k \) which it is defined with the below equation

\[ X^C_k = (X[f])_k^C; \quad \forall f \in C^0_{\square}. \quad (3.6) \]

Now we give the below theorem connected with the components of the complete lift of order \( k \) of \( X \).

**PROPOSITION 3.2.** Let \( X \) be a vector field defined on the vector bundle \( \square \) with form \( (3.1) \). In that case; the local expression of the complete lift \( X^C_k \) of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^C_k = \sum_{r=0}^{k} \{ (X^0_i)_k^C \frac{\partial}{\partial x^i} + (X^\alpha_0)_k^C \frac{\partial}{\partial u^\alpha} \}. \quad (3.7) \]

**PROOF:** Let \( X \) be a vector field defined on the vector bundle \( \square \) with form \( (3.1) \) and \( f \) be a function defined on the vector bundle \( \pi \). Then by considering Proposition 2.2. we write

\[ (X[f])_k^C = [X^0_i \frac{\partial f}{\partial x^i} + X^\alpha_0 \frac{\partial f}{\partial u^\alpha}]_k^C \]

\[ = \sum_{r=0}^{k} \{ (X^0_i)_k^C \frac{\partial}{\partial x^i} + (X^\alpha_0)_k^C \frac{\partial}{\partial u^\alpha} \}. \quad (3.7) \]

If the local expression of the complete lift \( X^C_k \) of \( k \) of \( X \) is
\[ X^C_k = X^r_i \frac{\partial}{\partial x^r_i} + X^r_{\alpha} \frac{\partial}{\partial u^\alpha} \quad (3.8) \]

with respect to adapted coordinates \{ x^r_i, u^\alpha : 0 \leq r \leq k \} in \( ^k E \). Then we write

\[ X^C_k \vert_{f^C_k} = \sum_{r=0}^{k} (X^r_i) X^C_{V^r_k} + X^r_{\alpha} X^C_{V^\alpha_k} \quad (3.9) \]

for all the function \( f \). Thus if we equalise the equation (3.7) with the equation (3.9) by considering the equation (3.6) then we obtain the below equations

\[ X^r_i = (X^r_i) (X_{0i})_{V^r_k} \quad \text{and} \quad X^r_{\alpha} = (X^r_{\alpha}) (X_{0\alpha})_{V^r_k} \quad 0 \leq r \leq k. \]

Finally, the local expression of the complete lift \( X^C_k \) of order \( k \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^C_k = \sum_{r=0}^{k} (X^r_i) (X_{0i})_{V^r_k} \frac{\partial}{\partial x^r_i} + X^r_{\alpha} (X_{0\alpha})_{V^r_k} \frac{\partial}{\partial u^{\alpha}}. \]

In addition to the complete-vertical lift of order \( (r,s) \) of a vector field \( X \) defined on the vector bundle \( \pi \) to its the extended vector bundle \( \pi^k \) can be defined for the integer numbers \( 0 \leq r \leq k \) and \( 0 \leq s \leq k-r \).

**DEFINITION 3.3.** Let \( X \) be a vector field defined on the vector bundle \( \pi \). In that case; the **complete-vertical lift of order \( (r,s) \)** of \( X \) to the extended vector bundle \( \pi^k \) is a vector field defined on \( \pi^k \) which is defined with the below equation

\[ X^{C^r_s}_{k} (f^{C^r_s}_{k}) = (X[f])^{C^r_s}_{k}; \quad \forall f \in 0 \quad \square \quad (3.10). \]

Now we give the below theorem connected with the components of the complete-vertical lift of order \( (r,s) \) of \( X \).

**PROPOSITION 3.3.** Let \( X \) be a vector field defined on the vector bundle \( \pi \) with form (3.1). In that case; the local expression of the complete lift \( X^{C^r_s}_{k} \) of order \( (r,s) \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^{C^r_s}_{k} = \sum_{t=0}^{k} (X^{0i}_t) (X_{0i})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial x^{r-t}_{V^s+k-t}} + (X^{0\alpha}_t) (X_{0\alpha})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial u^{\alpha}}. \]

**PROOF:** Let \( X \) be a vector field defined on the vector bundle \( \pi \) with form (3.1), and \( f \) be a function defined on the vector bundle \( \pi \). Then by considering Proposition 2.2, we write

\[ (X[f])^{C^r_s}_{k} = \sum_{t=0}^{k} (X^{0i}_t) (X_{0i})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial x^{r-t}_{V^s+k-t}} + (X^{0\alpha}_t) (X_{0\alpha})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial u^{\alpha}} \]

If the local expression of the complete-vertical lift \( X^{C^r_s}_{k} \) of \( k \) of \( X \) is

\[ X^{C^r_s}_{k} = X^i_t \frac{\partial}{\partial x^{i}_t} + X^{\alpha}_t \frac{\partial}{\partial u^{\alpha}} \quad (3.12) \]

with respect to adapted coordinates \{ \( x^r_i, u^\alpha \) : \( 0 \leq r \leq k \) \} in \( ^k E \). Then we write

\[ X^{C^r_s}_{k} \vert_{f^{C^r_s}_{k}} = \sum_{t=0}^{k} (X^i_t) (X_{0i})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial x^{r-t}_{V^s+k-t}} + (X^{\alpha}_t) (X_{0\alpha})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial u^{\alpha}} \quad (3.13) \]

for all the function \( f \). Thus if we equalise the equation (3.11) with the equation (3.13) by considering the equation (3.10) then we obtain the below equations

\[ X^{i}_t = (X^{i}_t) (X^{0i}_t)_{C^{r-t}_{V^s+k-t}} \quad \text{and} \quad X^{\alpha}_t = (X^{\alpha}_t) (X^{0\alpha}_t)_{C^{r-t}_{V^s+k-t}} \quad 0 \leq t \leq k \]

where because \( (X^{i}_t) = 0 \) for \( t \geq r \) integer numbers \( X^{i}_t = 0 \), \( X^{\alpha}_t = 0 \). Finally, the local expression of the complete-vertical lift \( X^{C^r_s}_{k} \) of order \( (r,s) \) of \( X \) to the extended vector bundle \( \pi^k \) is

\[ X^{C^r_s}_{k} = \sum_{t=0}^{k} (X^{0i}_t) (X_{0i})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial x^{r-t}_{V^s+k-t}} \quad \text{and} \quad (X^{0\alpha}_t) (X_{0\alpha})_{C^{r-t}_{V^s+k-t}} \frac{\partial}{\partial u^{\alpha}} \]

**PROPOSITION 3.4.** For all vector fields \( X, Y \in 3 \quad 0^1(\pi) \) and all function \( f \in 3 \quad 0^1(\pi) \)

\[ (X + Y)^{V^k}_k = X^{V^k}_k + Y^{V^k}_k \]

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ii) \((X + Y)C^k_k = X^C_k + Y^C_k\)

iii) \((fX)^C_k = f^k_k X^C_k\)

iv) \((fX)^C = \sum_{r=0}^{k} r^k_k f^k_{r-k} C^r_k X^{C^k_r}\)

v) \((fX)^C^{k_r} = \sum_{h=0}^{k_r} (\sum_{r=0}^{h} h^k_r f^k_{r-h} C^r_h) X^{C^h_{s-k}} 0 \leq r, s \leq k\)

\[ r+k \]

4. THE LIFTS OF 1-FORMS

Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\) and

\[ \omega = \omega_0^i dx^0 i + \omega_0^0 \alpha dx^0 \alpha \]  \quad (4.1)

be local expression of \(\omega\).

Thus if we define the vertical lift of a 1-form \(\omega\) defined on the vector bundle \(\pi\) to its the extended vector bundle \(\pi^k = (E, \pi^k, k, M)\) using the induction method for integer number \(k\), then:

**DEFINITION 4.1.** Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\). In that case; the vertical lift of order \(k\) of \(\omega\) to the extended vector bundle \(\pi^k\) is a 1-form defined on \(\pi^k\) which it is defined with the below equation

\[ \omega^k_k (X^k_k) = (\omega(X))^k_k; \quad \forall X \in J_0^1 (\pi) \]  \quad (4.2)

Now we give the below theorem connected with the components of the vertical lift of order \(k\) of \(\omega\).

**PROPOSITION 4.1.** Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\) with form (4.1). In that case; the local expression of the vertical lift \(\omega^k_k\) of order \(k\) of \(\omega\) to the extended vector bundle \(\pi^k\) is

\[ \omega^k_k = (\omega_0^i)^k_k dx^0 i + (\omega_0^0 \alpha)^k_k dx^0 \alpha. \]

**PROOF:** Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\) with form (4.1) and \(X\) be a vector field defined on the vector bundle \(\pi\). Then by considering Proposition 2.1, we write

\[ (\omega(X))^k_k = (\omega_0^i X^0 i + \omega_0^0 \alpha X^0 \alpha)^k_k \]

\[ (\omega(X))^k_k = (\omega_0^i X^0 i)^k_k + (\omega_0^0 \alpha)^k_k (X^0 \alpha)^k_k \]  \quad (4.3)

If the local expression of the vertical lift \(\omega^k_k\) of \(\omega\) is

\[ \omega^k_k = \omega_0^r dx^0 r + \omega_0^r \alpha dx^0 \alpha \]  \quad (4.4)

with respect to adapted coordinates \(\{ x^0 r, u^0 \alpha : 0 \leq r \leq k \}\) in \(E^k\). Then we write

\[ \omega^k_k (X^k_k) = \omega_0^r (x^r) (X^0 \alpha)^k_k + \omega_0^r (x^r) (X^0 \alpha)^k_k \]  \quad (4.5)

for all the vector field \(X\). Thus if we equalise the equation (4.3) with the equation (4.5) by considering the equation (4.2) then we obtain the below equations

\[ \omega_0^i (x^r)^k_k, \quad \omega_0^0 \alpha (x^r)^k_k, \quad \omega_0^i = 0, \quad \omega_0^r \alpha = 0 \quad 1 \leq r \leq k \]

Finally, the local expression of the vertical lift \(\omega^k_k\) of order \(k\) of \(\omega\) to the extended vector bundle \(\pi^k\) is

\[ \omega^k_k = \omega_0^i dx^0 i + \omega_0^r \alpha dx^0 \alpha. \]

Furthermore, we define the complete lift of order \(k\) of a 1-form \(\omega\) defined on the vector bundle \(\pi\) to its the extended vector bundle \(\pi^k = (E, \pi^k, k, M)\) by using the induction method for integer number \(k\), then:

**DEFINITION 4.2.** Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\). In that case; the complete lift of order \(k\) of \(\omega\) to the extended vector bundle \(\pi^k\) is a 1-form defined on \(\pi^k\) which it is defined with the below equation

\[ \omega^k_k (X^k_k) = (\omega(X))^k_k; \quad \forall X \in J_0^1 (\pi) \]  \quad (4.6)

Now we give the below theorem connected with the components of the complete lift of order \(k\) of \(\omega\).

**PROPOSITION 4.2.** Let \(\omega\) be a 1-form defined on the vector bundle \(\pi\) with form (4.1). In that case; the local expression of the complete lift \(\omega^k_k\) of order \(k\) of \(\omega\) to the extended vector bundle \(\pi^k\) is

\[ \omega^k_k = \sum_{r=0}^{k} \{ (\omega_0^i)^k_k x^0 r + (\omega_0^0 \alpha)^k_k x^0 \alpha \} \].
**PROOF:** Let \( \omega \) be a 1-form defined on the vector bundle \( \pi \) with form (4.1), and \( X \) be a vector field defined on the vector bundle \( \pi \). Then by considering Proposition 2.2, we write

\[
(\omega(X))_k^C = \{\omega_0 i X^0 d^0 + \omega_0 \alpha X^0 d^0 \} \alpha = \{\omega_0 i X^0 d^0 \} \alpha + \{\omega_0 \alpha X^0 d^0 \} \alpha \quad (4.7)
\]

\[
= \sum_{r=0}^{k} \{(X^0_0)\alpha C^V r d^r + (X^0_0)\alpha C^V r d^r \}
\]

\[
= \{(X^0_0)\alpha C^V r d^r \} \quad (4.8)
\]

If the local expression of the complete lift \( \omega_k^C \) of \( \omega \) is

\[
\omega_k^C \alpha = \omega_r d^r + \omega_r \alpha d^r \alpha \quad (4.8)
\]

Then by considering the equation (4.7) with equation (4.9) by considering the equation (4.6) then we obtain the below equations

\[
\omega_r \alpha = (\omega_0 \alpha) C^V r d^r \quad (4.10)
\]

Finally, the local expression of the complete lift \( \omega_k^C \) of \( \omega \) to the extended vector bundle \( \pi^k \) is

\[
\omega_k^C = \sum_{r=0}^{k} \{(X^0_0)\alpha C^V r d^r \}
\]

\[
\omega_k^C = \{(X^0_0)\alpha C^V r d^r \} \quad (4.11)
\]

Now we give the below theorem connected with the components of the complete-vertical lift of order \((r,s)\) of \( \omega \).

**PROPOSITION 4.3.** Let \( \omega \) be a 1-form defined on the vector bundle \( \pi \) with form (4.1). In that case; the local expression of the lift \( \omega_k^C \) of \( \omega \) to the extended vector bundle \( \pi^k \) is

\[
\omega_k^C \alpha = \sum_{r=0}^{k} \{(X^0_0)\alpha C^V r d^r \}
\]

\[
\omega_k^C = \{(X^0_0)\alpha C^V r d^r \} \quad (4.12)
\]

If the local expression of the complete-vertical lift \( \omega \) of \( \omega \) is

\[
\omega_k^C = \omega_r d^r + \omega_r \alpha d^r \alpha \quad (4.12)
\]

Finally, the local expression of the complete lift \( \omega_k^C \) of \( \omega \) to the extended vector bundle \( \pi^k \) is

\[
\omega_k^C = \sum_{r=0}^{k} \{(X^0_0)\alpha C^V r d^r \}
\]

\[
\omega_k^C = \{(X^0_0)\alpha C^V r d^r \} \quad (4.13)
\]

for all the vector field \( X \). Thus if we equalise the equation (4.7) with equation (4.9) by considering the equation (4.6) then we obtain the below equations

\[
\omega_r \alpha = (\omega_0 \alpha) C^V r d^r \quad (4.10)
\]

**DEFINITION 4.3.** Let \( \omega \) be a 1-form defined on the vector bundle \( \pi \). In that case; the complete-vertical lift of order \((r,s)\) of \( \omega \) to the extended vector bundle \( \pi^k \) is a 1-form defined on \( \pi^k \) which it is defined with the below equation

\[
\omega_k^C \alpha = \{(X^0_0)\alpha C^V r d^r \} \quad (4.14)
\]

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where because \((r')_t = 0\) for \(t > r\) integer numbers \(\omega_{ti} = 0\), \(\omega_{t\alpha} = 0\). Finally, the local expression of the complete-vertical lift \(\omega_{kV^s}^C \) of order \((r,s)\) of \(\omega\) to the extended vector bundle \(k\) is
\[
\omega_{kV^s}^C = \sum_{i=0}^{k} \left\{ \begin{array}{l}
(\omega_{f_i})_k C^{r-t} \text{d}x^i + \binom{r}{k} \omega_{f_v} \nabla^{r-t} \text{d}u^\alpha \end{array} \right. \]

**PROPOSITION 4.4.** For all 1-forms \(\omega, \theta \in \mathfrak{F}_1^0(\pi)\) and all function \(f \in \mathfrak{F}_0^0(\pi)\)

i) \((\omega + 0)_k V^k = \omega_k + \theta_k V^k\)

ii) \((\omega + 0)_k C^k = \omega_k + \theta_k C^k\)

iii) \((f\omega)_k V^k = f_k \omega_k V^k\)

iv) \((f\omega)_k C^k = \sum_{r=0}^{k} \binom{k}{r} f_k V^{s-r} C^r \omega_k C^{k-h} V^r\)

v) \((f\omega)_k V^s = \sum_{h=0}^{r} \binom{r}{h} f_k V^{s+h} C^{r-h} \omega_k C^{k-h} V^r\) \(0 \leq r, s \leq k\) \((r+s=k)\).

4. CONCLUSION

This study is the generalising study of the complete-vertical lift of the differential elements defined on the extended vector bundles. In addition to the previous studies connected with the vertical and complete lift of differential elements are special cases of this study.

REFERENCES