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A Study of a Non-linear Reaction Diffusion Equation Representing Initial and Boundary Value Problems by LDTM

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Abstract*:* In this paper, we study the exact solutions of non-linear Reaction-Diffusion equation using the Laplace-Differential Transform method (LDTM). We apply Laplace transform in time domain and differential transform in space domain using initial and boundary conditions. We find that this method requires straightforward differentiation and a few elementary operations for the solution unlike other typical methods which requires integration. Illustrative examples are presented to demonstrate the applicability and efficiency of the technique. The concluding results are accurate and with less computational effort than some existing studies.

Keywords*:* LDTM, Non-linear Reaction-Diffusion equation, Initial Conditions, Boundary conditions.

1. INTRODUCTION

In engineering, several processes are presented by boundary value problems. In general, classical methods fail to produce good approximation for boundary value problems. In this paper, LDTM which is a combination of differential transform method and Laplace transform method are applied for solving IVPs and BVPs arising in the reaction diffusion equation. Reaction-diffusion equation was first introduced by KPP (Kolmogorov-Petrovsky-Piskounov). Jafari and Gejji (2006) have introduced Adomian decomposition method for solving linear and nonlinear fraction diffusion equation and wave equations. Ray and Bera (2006) have found the analytical solution of a fractional diffusion equation by Adomian decomposition method. Kandilarov (2007) has used immersed interface method for a system of linear reaction-diffusion equations with non-linear singular own sources. Othman and Mahdy (2010) have applied the DTM and Variation iteration method to solve Cauchy reaction-diffusion problems. Bhadauria *et* al. (2011) have solved reaction diffusion equation using differential transformation method. Shen et al. (2011) have used the Relaxation method for unsteady convection-diffusion equations. Gupta and Verma (2014) have studied the Cauchy reaction diffusion equation.

In this paper, we are interested in the LDTM to solve non-linear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + pu^2 + qu + z(x,t); \quad (x,t) \in \Omega \subset R^2 \qquad (1.1)$$

subject to initial condition

$$u(x,0) = f(x), \quad x \in R, \qquad (1.2)$$

and the Dirichlet boundary conditions

$$u(0,t) = g_1(t),\ u(1,t) = g_2(t), \quad t \in R \qquad (1.3)$$

or the Neumann boundary conditions

$$u(0,t) = g_1(t),\ u_x(1,t) = g_3(t), \quad t \in R \qquad (1.4)$$

The DTM is iterative procedure for obtaining analytic Taylor's series solution of differential equations. It gives a truncated series solution; this series gives a good approximation to the true solution in a very small region. The algorithm of DTM was first proposed by J.K. Zhou in 1986, its main application concern with both linear and non-linear initial value problems in electric circuit analysis. Wazwaz (2001) has discussed the numerical solution of sixth-order boundary value problems by using modified decomposition method. Hassan (2004) has solved higher order boundary value problems by DTM. Bor-Lih (2004) discussed the thermal boundary-layer problems in a semi-infinite flat plate by the DTM. Gupta (2011) has investigated the reduced differential transform method and the homotopy perturbation method to find the approximate analytical solutions of fractional Benney–Lin equation.

Recently, Alquran et al. (2012) have introduced the LDTM for solving linear non-homogeneous PDEs with variable coefficients.

The Laplace transform method can be regarded as a finite difference and a DTM is a numerical method based on the Taylor series expansion, which constructs on analytical solution in the form of a polynomial. The main advantage of LDTM is that it can be applied directly to linear and non-linear PDEs without requiring linearization, discretization or perturbation. Therefore, it is not affected

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by errors associated to discretization. The results obtained are in good agreement with those obtained numerically.

2. DIFFERENTIAL TRANSFORMATION METHOD

The one variable differential transform [8] of a function $u(x,t)$ is defined as

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} \quad ;k \geq 0 \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse of one variable differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x-x_0)^k, \tag{2.2}$$

where x_0 is the initial point for the given condition. Then the function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k. \tag{2.3}$$

3. SOLUTION OF THE PROBLEM BY LDTM

To illustrate the basic idea of Laplace differential transform method [7], we consider the following one-dimensional time dependent reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + pu^2 + qu + z(x,t), \tag{3.1}$$

$$(x,t) \in \Omega \subset R^2$$

subject to initial condition

$$u(x,0) = f(x), \quad x \in R, \tag{3.2}$$

and the Dirichlet boundary conditions

$$u(0,t) = g_1(t),\ u(1,t) = g_2(t), \quad t \in R \tag{3.3}$$

or the Neumann boundary conditions

$$u(0,t) = g_1(t),\ u_x(1,t) = g_3(t), \quad t \in R \tag{3.4}$$

Taking the Laplace transform on (3.1), can be obtain

$$sL[u(x,t)] - u(x,0) = L\left[D\frac{\partial^2 u}{\partial x^2} + pu^2 + qu + z(x,t)\right].$$

Using initial condition from equation (3.2), we get

$$L[u(x,t)] = \frac{f(x)}{s} + \frac{1}{s}L\left[D\frac{\partial^2 u}{\partial x^2} + pu^2 + qu + z(x,t)\right].$$

Applying Inverse Laplace transform on both sides

$$u(x,t) = f(x) + L^{-1}\left[\frac{1}{s}L\left[D\frac{\partial^2 u}{\partial x^2} + pu^2 + qu + z(x,t)\right]\right].$$

Now taking Differential transform on both sides w.r.to ‘x’, we get

$$U_k(t) = F(k) + L^{-1}\left[\frac{1}{s}L\left[D(k+2)(k+1)U_{k+2}(t)\right]\right] + L^{-1}\left[\frac{1}{s}L\left[p\sum_{r=0}^{k}U_r(t)U_{k-r}(t) + qU_k(t) + Z(k)\right]\right]. \tag{3.5}$$

Now, apply the differential transform method on the given Dirihlet and Neumann boundary conditions (3.3) and (3.4), we get

$$U_0(t) = g_1(t). \tag{3.6}$$

Let us assume

$$U_1(t) = a\,h(t). \tag{3.7}$$

By the definition of DTM, we take

$$u(1,t) = \sum_{i=0}^{\infty} U_i(t), \quad u_x(1,t) = \sum_{i=0}^{\infty} iU_i(t). \tag{3.8}$$

By equation (3.8), we calculate the value of ‘a’.

Now by the above equations (3.6) and (3.7) in (3.5), by straightforward iterative steps, we obtain $U_k(t)(k = 2,3,.......)$.

Then the inverse transformation of the set of values $U_k(t)(k = 0,1,2,3,.......)$ gives approximate solution as,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k.$$

Which is the closed form of the solution.

4. NUMERICAL EXAMPLES

To illustrate the applicability of LDTM, we have applied it to non-linear PDEs which are homogeneous as well as non-homogeneous.

Example 4.1: Consider the following non-linear PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 - x^2t^2 + x, \tag{4.1}$$

with initial condition

$$u(x,0) = 0, \tag{4.2}$$

and

$$u(0,t) = 0, u_x(0,t) = t. \tag{4.3}$$

The exact solution can be expressed as:

$$u(x,t) = xt.$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to ‘t’ on eqn. (4.1), we get

$$sL[u(x,t)] - u(x,0) = L\left[\frac{\partial^2 u}{\partial x^2} + u^2 - x^2t^2 + x\right].$$

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By using initial conditions from equation (4.2), we get

$$L[u(x,t)]=\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u^2-x^2t^2+x\right].$$

Here, we applying the inverse Laplace transformation with respect to 's', on both sides:

$$u(x,t)=xt-\frac{x^2t^3}{3}+L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u^2\right]\right].$$

Now applying the DTM with respect to space variable 'x', we get

$$U_k(t)=t\delta(k-1,t)-\frac{t^3}{3}\delta(k-2,t)$$
$$+L^{-1}\left[\frac{1}{s}L\left[(r+2)(r+1)U_{r+2}(t)+\sum_{r=0}^{k}U_r(t)U_{k-r}(t)\right]\right], \quad (4.4)$$

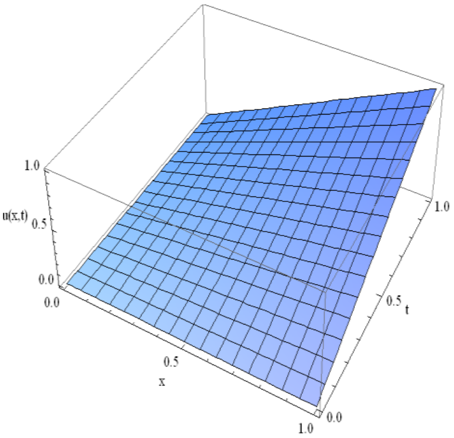
from the initial condition given by Eq. (4.3), we have

$$U_0(t)=0,\ U_1(t)=t. \quad (4.5)$$

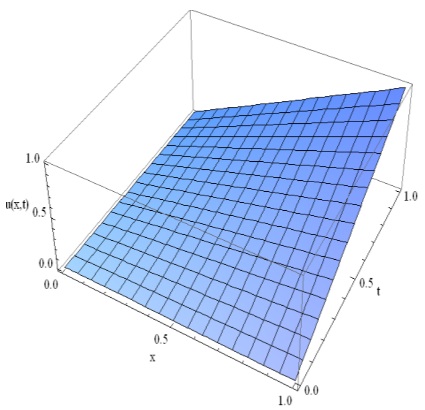
Substituting (4.5) into (4.4) and by straightforward iterative steps and we get the component $U_k(t)$, $k\geq 0$ of the DTM can be obtained. When we substitute all values of $U_k(t)$ into equation (3.11), then the series solution can be formed as

$$u(x,t)=xt.$$

which is the exact solution.



(a)



(b)

Fig. 1: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.2.

Fig. 1(b) represents the LDTM solution of $u(x,t)$ for the problem (1) from *t=0* to *1* and *x=0* to *1* and it is linearly increases with time *t* which is similar to exact solution Fig. 1(a). The accuracy of the LDTM for nonlinear initial value problem is controllable and absolute errors are very small with the choice of *t* and *x*. The result also obtains by mathematica-8 and both the results comparing through the Fig. 1(a) and 1(b). There are no visible differences in the two solutions.

Example 4.2: Consider the following non-linear PDE

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u^2-u-t^2e^{2x}+e^x, \quad (4.6)$$

subject to the initial conditions

$$u(x,0)=0, \quad (4.7)$$

and the Dirichlet boundary conditions

$$u(0,t)=t, \qquad u(1,t)=t\,e. \quad (4.8)$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to 't' on equation (4.6), we get

$$sL[u(x,t)]-u(x,0)$$
$$=L\left[\frac{\partial^2 u(x,t)}{\partial x^2}+u^2(x,t)-u(x,t)-t^2e^{2x}+e^x\right].$$

By using initial conditions from equation (4.7), we get

$$L[u(x,t)]=\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u^2-u-t^2e^{2x}+e^x\right].$$

Here, we applying the Inverse Laplace transformation w.r.t. 's' on both sides, and we get

$$u(x,t)=te^x-\frac{t^3}{3}e^{2x}+L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2}+u^2-u\right]\right].$$

Now applying the Differential transformation method with respect to space variable 'x', we get

$$U_k(t)=t\frac{1}{k!}-\frac{t^3 2^k}{3k!}+L^{-1}\left[\frac{1}{s}L[(k+2)(k+1)U_{k+2}(t)]\right]$$
$$+L^{-1}\left[\frac{1}{s}L\left[+\sum_{r=0}^{k}U_r(t)U_{k-r}(t)-U_k(t)\right]\right], \quad (4.9)$$

From the initial condition given by Eq. (4.8), we have

$$U_0(t)=t. \quad (4.10)$$

And let us assume

$$U_1(t)=at. \quad (4.11)$$

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Substituting (4.10) and (4.11) into (4.9) and by straightforward iterative steps, we obtain

$$U_2(t) = \frac{t}{2}, U_3(t) = \frac{a + at - 1 - 2at^2 + 2t^2}{3!},$$

$$U_4(t) = \frac{2t^2 + t - 2a^2t^2}{4!}, \ldots \qquad (4.12)$$

From equation (3.10), we take

$$u(1,t) = \sum_{i=0}^{\infty} U_i(t) = t\,e,$$

and, we get

$a = 1.$

Substituting the value of a into equations (4.11) and (4.12), now by straightforward iterative steps, we get the component $U_k(t),\ k \geq 0$ of the LDTM can be obtained. When we substitute all values of $U_k(t)$ into equation (3.11), then the series solution can be formed as

$u(x,t) = t\,e^x.$

which is the exact solution.

(a)

(b)

Fig. 2: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.4.

Fig. 2(b) represents the LDTM solution of $u(x,t)$ for the problem (2) from *t=0* to *1* and *x=0* to *1* and it is linearly increases with time *t* which is similar to exact solution Fig. 2(a). The accuracy of the LDTM for nonlinear initial value problem is controllable and absolute errors are very small with the choice of *t* and *x*. The result also obtains by mathematica-8 and both the results comparing through the Fig. 2(a) and 2(b). There are no visible differences in the two solutions.

5. CONCLUSION

In this paper, the new algorithm of the Laplace differential transform method has been presented with an aim to investigate the conditions to converge the solutions of the non-linear Reaction-diffusion equations. The presented method is applied to solve initial or boundary value problem. The LDTM gives approximations of higher accuracy and quantitatively reliable results for closed form solutions if existing.

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