

Mathematical Modeling of Social Networks Using Graphical Theory and Network Centrality Measures

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Abstract: Social Network Analysis (SNA) is an interdisciplinary approach for studying relationships, interactions, and information flows among individuals, groups, organizations, and other entities. A social network can be represented mathematically as a graph consisting of nodes (vertices) and links (edges), where nodes represent users or entities and edges represent social relationships such as friendship, collaboration, communication, or following. The rapid growth of social networking platforms such as Facebook, Twitter, Instagram, and other Web 2.0 applications has increased the importance of graph-based approaches for understanding online social interactions. Social graphs and labelled graphs provide useful representations of relationships and associated information. Graph-based community clustering can identify groups of users sharing common interests, behaviours, backgrounds, or preferences. Graph theory also helps analyse information diffusion by representing the movement of information from senders to receivers through different communication channels. Important network measures such as degree centrality help identify influential or highly connected nodes, while modularity measures the strength and quality of community structures by comparing actual connections with those expected by chance. These techniques have practical applications in business communication, recommendation systems, network optimization, and law enforcement. Thus, graph theory and SNA provide powerful mathematical and visual tools for understanding the structure, behaviour, and dynamics of modern social networks.

Keywords: SNA-Social Network Analysis, Communication, Nodes, links (edges), Diffusion, Optimization

INTRODUCTION

Graph theory is an old subject, but one that has many fascinating modern applications. Graph theoretical ideas are highly utilized by computer science applications. Especially in research areas of computer science such as data mining, image segmentation, clustering, image capturing, networking. These applications in turn have offered important stimulus to the development of the field, leading to generalizations of important graph theoretical concepts and challenging questions about them. The powerful combinatorial methods found in graph theory have also been used to prove significant and well-known results in a variety of areas in mathematics itself. Modeling of network topologies can be done using graph concepts. In the same way the most important concept of graph coloring is utilized.

These days, networks and talk about “Social Networks” is everywhere. It was not always so, the discipline of social network analysis developed quietly during the 1960s and 1970s out of scattered lines of work in anthropology, sociology, psychology, mathematics, and communication. It was not until the 1970s that a formal group was founded to study social networks. Today network science is an interdisciplinary fields uniting physics, computer science, biology, brain science, sociology, anthropology, economics and a variety of other fields. All are come tighter around the idea of a “Network” So what is a network ? minimally a network as composed of a set of units or entities. These are sometimes called nodes or vertices, however what makes these units into a network is the fact that at least some pairs of them are joined by asset of links or ties. These are also sometimes called edges. So a network is essentially a set of nodes some of which are linked together. In social networks the nodes can be people and the connections can be any type of social link (e.g., friendship, enmity, co-working), whether positive or negative, between those people.

1.1. METHODOLOGY: The methodology for studying Social Network Analysis (SNA) using **graphical theory** can be organized into the following steps:

1.2. SOCIAL NETWORK:

After the search engine, social network is another hot web application with involves the user as the main actor. The success of social networking websites such as Facebook, myspace, Twitter, Instagram and others, shows the revolution caused by web 2.0. Users have their “second life” on the web, which is a virtual environment, to meet friends, discuss opinions, play online games and share information. It also is more and more common that user are members of more than one social network at a time and their friends similarly use multiple social networks.

1.3. SOCIAL GRAPH:

The social graph is a graph that represents social relations between entities. In short, it is a model or representation of a social network, where the word graph has been taken from graph theory.

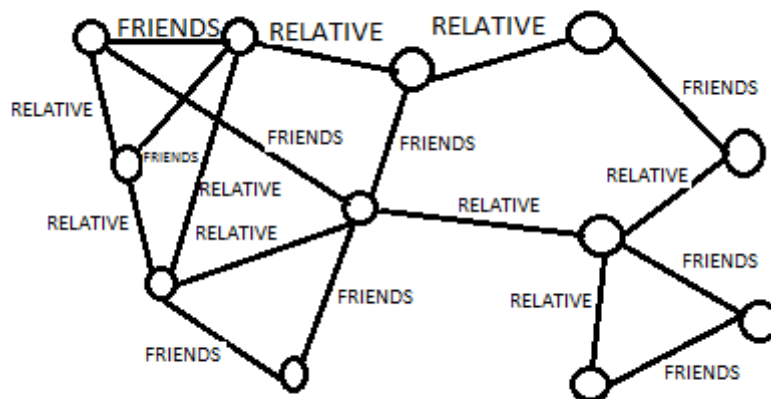
For Example:-

In the above graph each person is represented by a dot called a node and the friendship relationship is represented by a line called edge.

1.4. SOCIAL NETWORK GRAPH:

A Social network graph is a graph where the nodes represent people and the lines between nodes called edges, represent Social connections between them, such as friendship or working together on a project. These graphs can be either undirected or directed.

Often, Social Graphs are undirected, as for the facebook friends graph. But they can be directed graphs, as for example the graphs of followers on twitter and google +.



SOCIAL NETWORK GRAPH

1.5. SOCIAL NETWORK DIAGRAM:

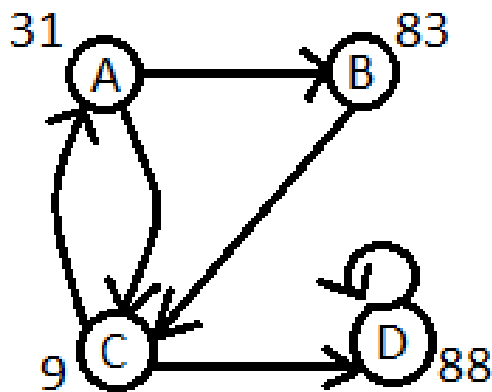
A social network diagram visually displays the relationships and interactions between people, groups, computers and other information entities. It maps out the nodes (individuals or groups) and the links (relationships or interactions) that connect them.

The Social network users' graph can be either vertex labelled or edge labelled.

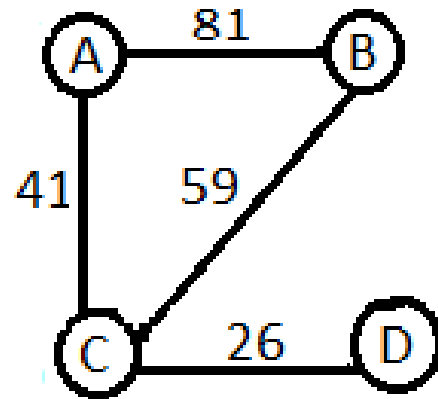
1.6. LABELLED GRAPH:

If the vertices and edges of a graph are labelled with name, data or weight then it is called a labelled graph.

If the vertices are labelled with name or data then it is called vertex labelled or if edges are labelled with names or data then it is called edge labelled.



Directed Graph with labeled Vertices



Undirected Graph with Labelled Edges

Applications of graph theory in a social network include community clustering, information diffusion, influence modeling and trust modeling etc.

1.6.USE OF GRAPH FOR COMMUNITY CLUSTERING IN A SN (SOCIAL NETWORK):

A community means a group of people or users and clustering is the task of dividing the population or data points into a number of groups based on some common properties.

Thus community clustering in social network is defined as the groups of people or users distinguished with some common properties that are common interest or hobbies or locality backgrounds.

For example, in a community a node represents user have similarities with other users of that community.

In addition, user in a social network (SN) can be placed in different communities based on their interest or behaviour or background or preferences. Users belongs to one community having similar behaviours or attitudes towards some given event or topic.

In addition, user communities can be formed based on the activities in an online social networking such as commenting on the similar topic and posting similar contents on the social networking sites.

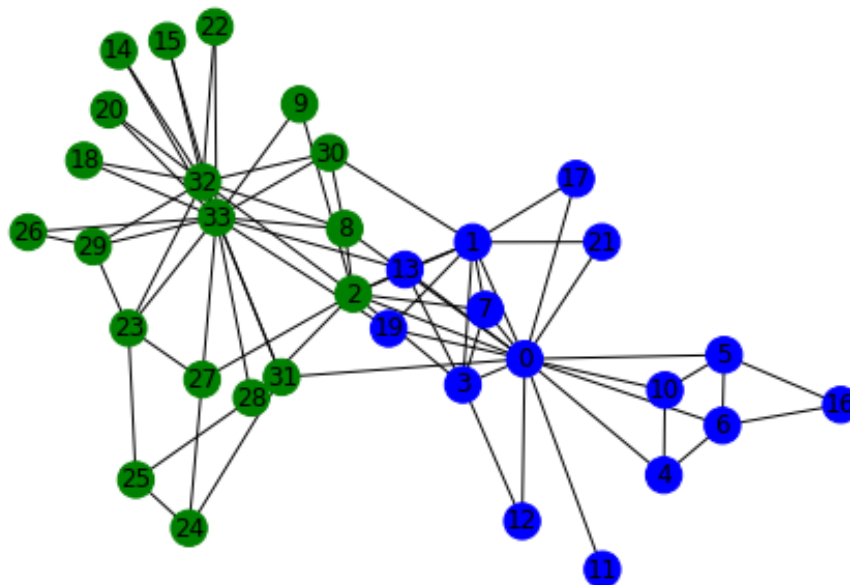


FIG: Two Communities in a Graph

1.7.USE OF GRAPHS FOR INFORMATION DIFFUSION IN SOCIAL NETWORKS:

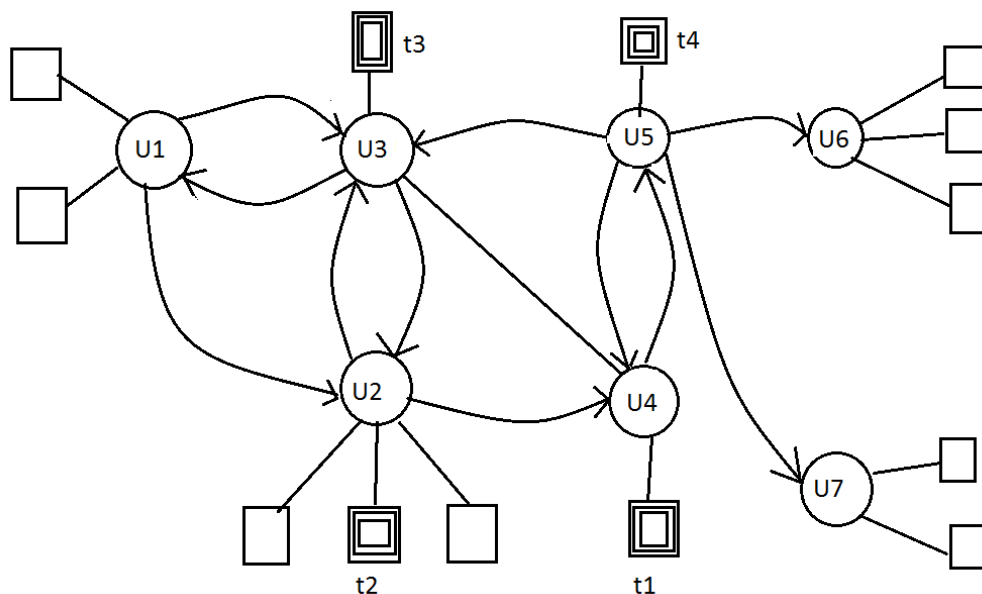
Diffusion is the process by which information is spread from one place to another through interactions. The diffusion process involves three main elements as follows:

1.Sender - A sender (or a group of senders) is responsible for initiating the diffusion process.

2.Receiver - A receiver (or a group of receivers) receives the diffusion information from the sender. Commonly the number of receivers is higher than the number of senders.

3.Medium - This is the channel through which the diffusion information is sent from the sender to the receiver. This can be TV, newspaper, social media, (example a tweet on Twitter.)

For example the diffusion process of a particular information I in a social network showing by the following figure that red nodes took part in the diffusion process.



The information I sending from T₂, T₃, T₁, and T₄ to U₂, U₃, U₄, U₅ respectively.

In some cases the users communities are formed based on the trust/interest/ similarities and information diffusion can be done with the relevant communities. The three communities formed based on the interest are shown in figure a with the help of these communities, the diffusion process can become more strong and information can reach the users.

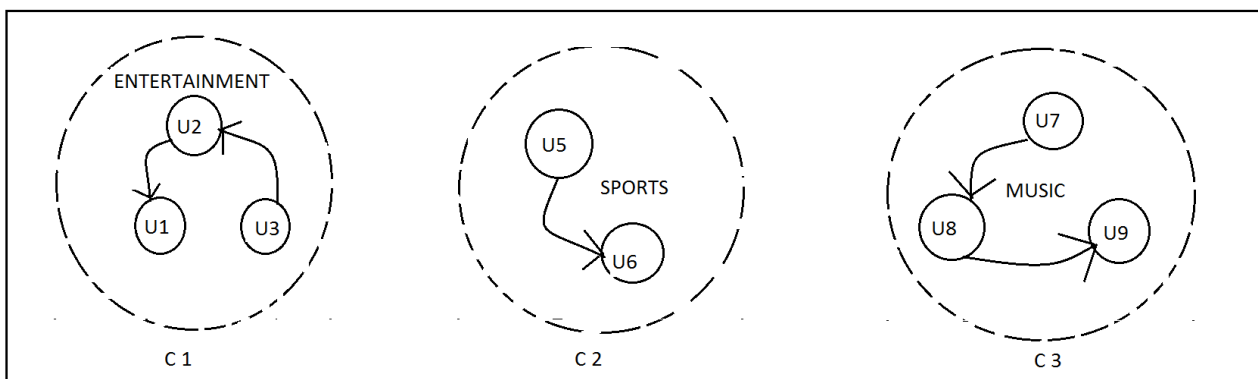
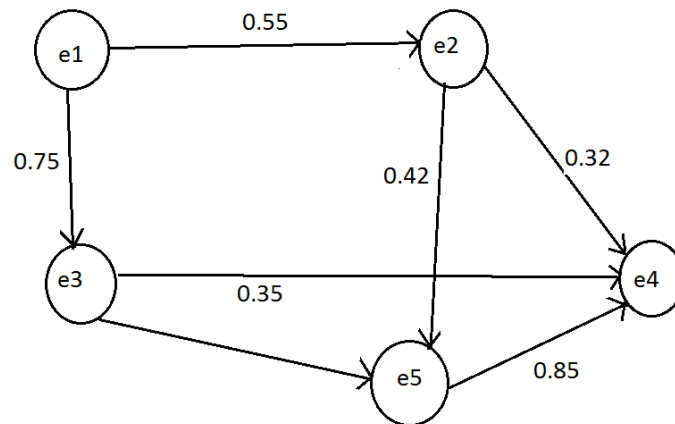


Fig:- Information diffusion in a social networking w.r.t. communities

2.1. Users influence / trust score representation in social networks.

Graph theory concepts can be applied to represent the users trust on each other.

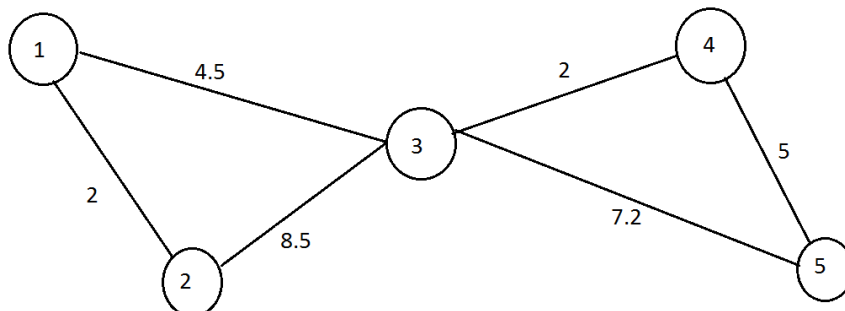
For example.



In the above figure, the nodes represent the experts and edges represent the trust of each expert on each other. Here graph is used known as edge labelled graph.

2.2. Similarity Modeling / Representing Between Social Network Users

The graph theory can be used to show the similarity / dissimilarities between users. The users can be grouped based on different similarities such as location, behaviours and background. The dissimilarities graph among five social networks is shown in the given figure;



3.SOCIAL NETWORK ANALYSIS (SNA)

Social Network Analysis is the mapping and measuring of relationships and flows between people, groups, organizations, computers and other connected information / knowledge entities. The nodes in the network are the people and groups while the link show relationships or flows between the nodes. Social Network Analysis provides both a visual and a mathematical analysis of human relationships.

4.1. PRACTICAL APPLICATIONS:

Businesses use Social Network Analysis to analyze and improve communication flow in their organization, or with their networks of partners and customers.

Law enforcement agencies use Social Network Analysis to identify criminal and terrorist networks from traces of communication that they collect; and then identify key players in these networks.

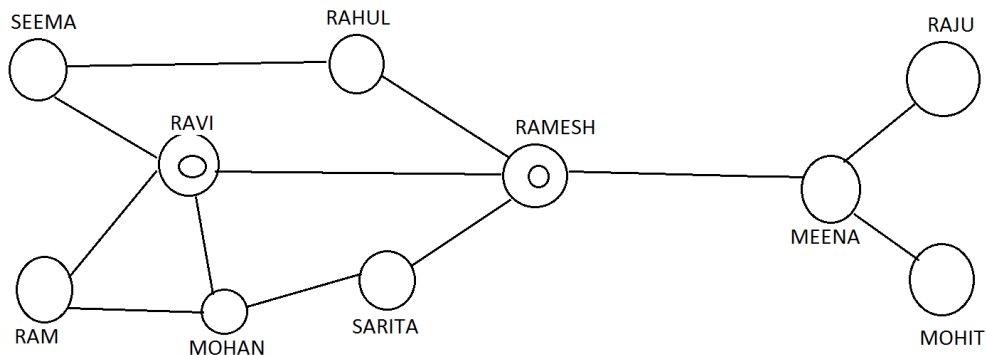
Social network sites like Facebook use basic element of Social Network Analysis to identify and recommend potential friends based on friends of friends.

Network operators (telephony, cable, mobile) use Social Network Analysis like methods to optimize the structure and capacity of their networks.

Now, we define four important matrices related to social network analysis, and show them graphically.

4.2. Degree Centrality:

This is simply the number of edges of the edge. The more edges, relatively speaking within the graph, the more important a node. The node with higher edges typically looks like a “hub” of activity if you were to visualize the graph.

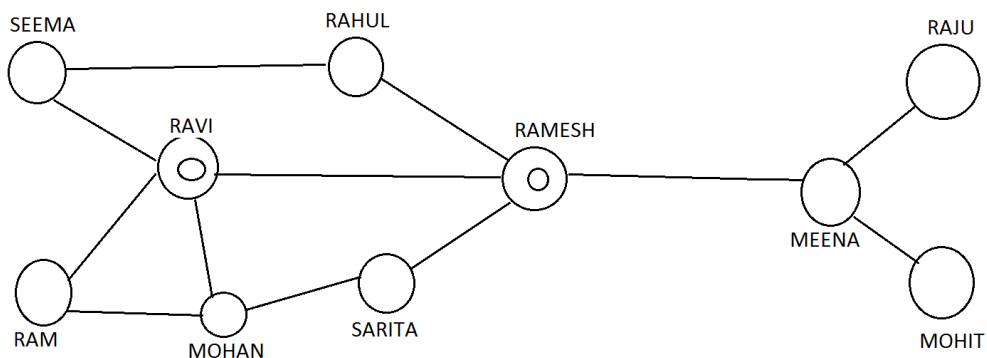


- Moderate degrees centrality
- High degree centrality

In our above network diagram, Ravi has the highest degree centrality which means that she is more active in a network compared to all the other users.

4.3. Closeness centrality:

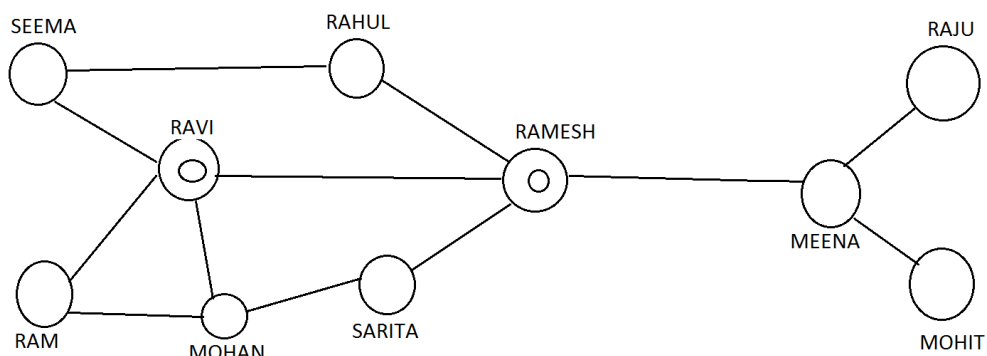
This measure is the inverse of the sum of all shortest path to other vertices. Closeness can be regarded as a measure of how fast it will take to spread information to all other nodes. If a node has strong closeness centrality, it is in a position, with its relationship, to spread information quickly. These people (if nodes are people in the graph) can be important influencers of the network.



- Moderate closeness centrality
- High closeness centrality

4.4. Betweenness Centrality:

As an indicator of how many paths a node is on, it's betweenness centrality represent that nodes ability to make connections to other groups in the graph.

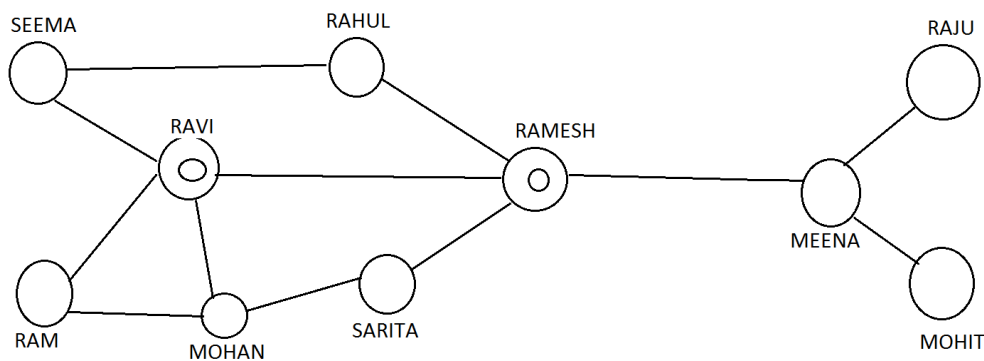


- High betweenness
- Moderate betweenness

In the presented example, Ramesh has the highest between centrality because he is between Ravi and Meena, who are between other users. Ravi and Meena have a slightly lower betweenness because they are essentially only between their own groups.

4.5. Eigenvalue centrality:

Finally, there is eigenvalue centrality, which assigns scores to all nodes in the network that indicate the importance of the node in a graph. Scores are assigned based on the principle that high scoring nodes contribute more to the score than equal connection to low scoring vertices. eigen values centrality useful in determining who is connected to the most connected nodes.



- High Eigenvalues centrality
- Moderate Eigenvalues centrality

In the above network diagram it can be observed that Ravi and Ramesh are closer to other highly close entities in a network. Rahul and Mohan are also highly close, but to a lesser value. In addition graph theory concepts can be applied to show the minimum number of connections that can be made by different number of users in a social network. The edges represent the connection with different users, and vertices represent the actual users.

5.1. Interpretation of Measures:

Centrality Measure	Interpretations in Social Network
Degree	How many People can this person reach directly ?
Betweenness	How likely is the person to be the most direct route between two people in the network ?
Closeness	How fast can this person reach everyone in the network ?
Eigenvalue	How well is this person connected to other well connected people ?

5.2. Analyzing the modularity in a social network user's graph:-

The nodes of many real-world networks are often more strongly connected with some sets of nodes than with others, giving rise to densely linked communities of nodes called modules.

Social networks are generally large scale networks that comprises of millions or billions of nodes. One of the ways to analyze the properties of these nodes is by understanding their group behaviour (community properties). However detection of communities in such a largescale network is very tedious and requires matrices to understand the effectiveness of the community detection approach.

Modularity is a metric that measure the goodness of communities.

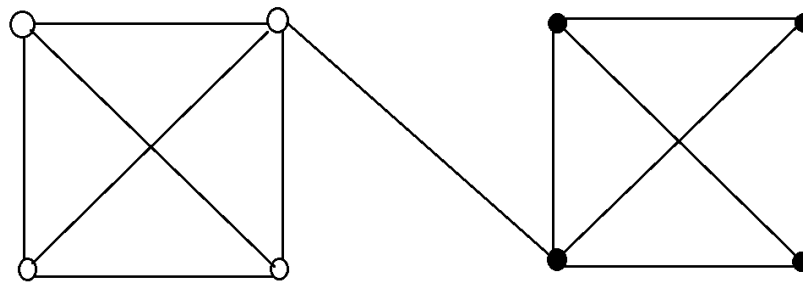
Further, a community is actually a group of nodes that are well connected among themselves which represent the edge connections are not by random.

So, modularity checks how better the connection in a community are then if those connections were made randomly.

In short, modularity is one measure of the structure of networks or graphs. Networks with higher modularity have dense connections between the nodes within modules.

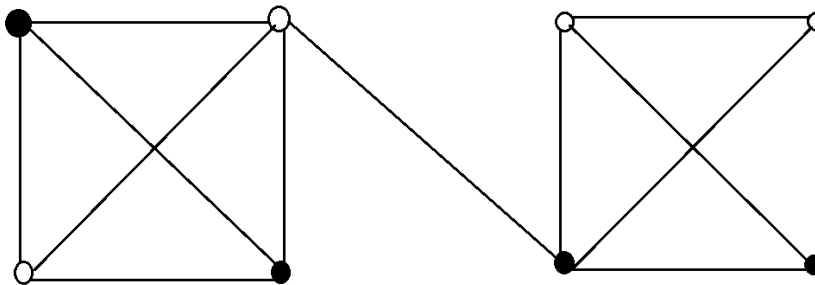
Modularity is widely used as a measure for how good a clustering is.

5.3.High modularity: more edges within module that you expect by chance.



High Modularity

The connection between the nodes are more.



Low Modularity

5.4. User identification across social networks by analyzing structural properties of the graphs.

User identification, which is also termed as user recognition, user identity resolution, user matching, and anchor linking, aims to find identical users among different social network platforms.

Social networking is an interactive internet of things. The symmetry of the network can reflect the similar friendships of user on different social networks. A user's behaviour habits are not easy to change, and users usually have the same or similar display names and published contents among multiple social networks.

Therefore, the symmetry concept can be used to analyze the information generated by the user for user identification. User identification plays a key role in building better information about social network user profiles.

The widespread application of social networks however, also brings many problems, such as privacy leaks, hacker attacks, and network security. Because there is a certain connection between users on the Social Network and the information generated by the user (profile data, behaviour data, network structure) has certain symmetry with this feature, we can identify the entity user behind multiple social network accounts. Display names and user published.

Contents are very easy to obtain compared to other type of users data across different social networks.

User identification method based on user behaviour habits, user profile data based & network structure based, and user content based.

In user identification method based on user behaviour habits, we analyze the users' personalized naming habits in terms of display names, then utilize different similarity calculation methods to measure the similarity of the features contained in the display names.

6.1.User Profile Data Based User Identification:

User profile data refers to data that a user needs to enter or select when registering a Social Network account and include their username, gender, birthday, education etc.

In the process of filling out personal information, users are likely to refer to their previous registration information on other social networks. Studies show that users have similarities in profile data on different social networks, that is, there is symmetry in such data.

Here, for user identification, we analyzed the user's personal information on different social networks, represented it as a set of words, and then calculated the similarity between two social accounts.

6.2. Network Structure Based User Identification:

User identification based on network topology information refers to methods in which the friend relationships between users are treated as equivalent to the network topology, allowing similarity matching between nodes to be performed.

The network topology formed by users has certain symmetry on different social networks.

Since the friend relationship formed by the user is basically fixed, identifying user on different social networks can be completed according to the number of shared identified friends.

6.3. User Generated Content Based User Identification:

User identification based on user generated data focuses primarily on the content published by users. Generally speaking when users post social contents they usually synchronize them with the other social networks they hold. We can use the principle of symmetry to respectively analyze the data generated by users on different social network, such as Geographic locations, tags and status timestamps.

We introduced a method of across social network user identification based on display name and interest graph.

Given to social networks S_1 and S_2 , U_1^1 & G_1^2 denote a set of display names and interest graph on social network S_1 and $U_1^{n1} \in C_1^1$, $U_1^{n2} \in G_1^2$ where U_1^{n1} and U_1^{n2} denote the display names and interest graph of the n^{th} registered user in S_1 .

User identification is the mining of physical users behind multiple different social network accounts.

In other words, we need to determine whether (V_1^n, V_2^k) is the same user and whether (V_1^m, V_2^k) is a different user.

We define E_{ij} as the identified account pair. If $(V_1^n, V_2^k) \in E_{ij}$ this proves that the account pair (U_1^n, U_2^k) belongs to the same user in real life.

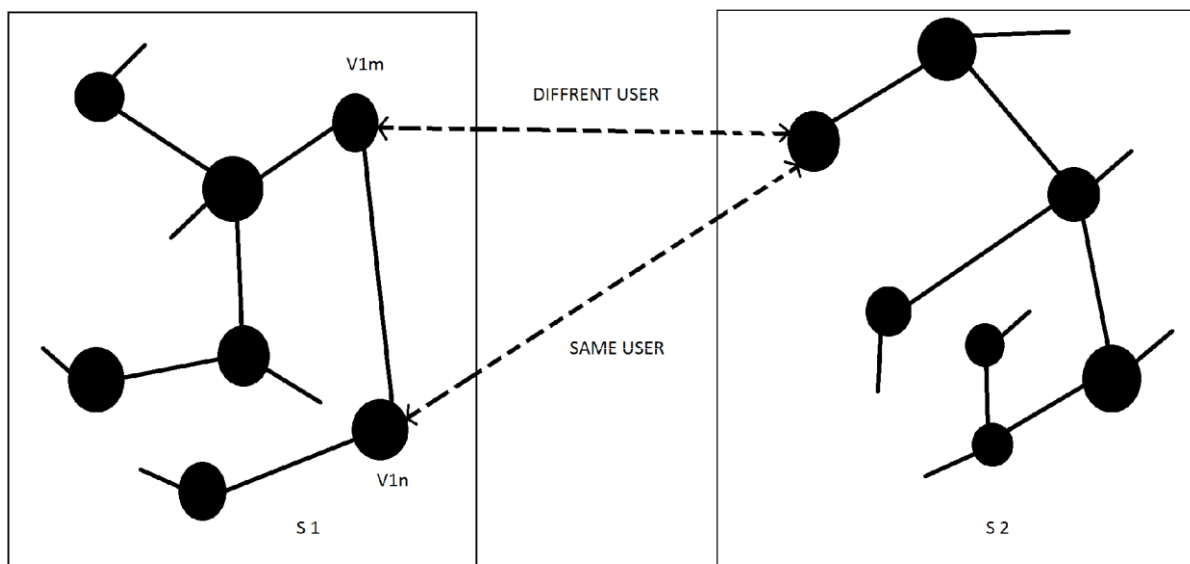


Fig: Example of across social networks user identification

7. SOCIAL – ATTRIBUTE NETWORK (SAN) MODELLING AND ANALYSIS BY A GRAPH

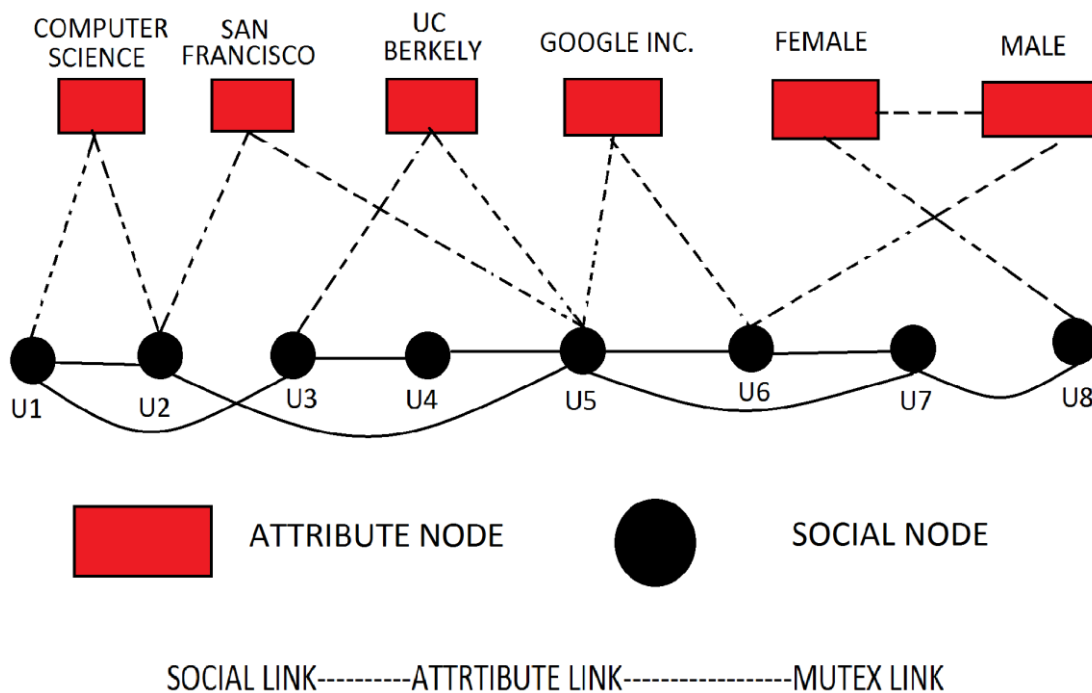
Given a social network G with M distinct categorical attributes, an attribute matrix A and mutex attributes Set L , we create an augmented network by adding M additional nodes to G , with each additional node corresponding to an attribute. For each node u in G with positive or negative attribute a , we create an undirected link between u and a in the augmented network. For each mutually exclusive attribute pair (a,b) , we create an undirected link between a and b . we call this augmented network the social attribute network (SNA) Since it includes the original social network interactions, relations between nodes and their attributes and mutex links between attributes.

Nodes in the SAN model corresponding to nodes in G are called social nodes, while nodes representing attributes are called attributes nodes. Links between social nodes are called social links and links between social nodes and attribute nodes are called attribute links. Attribute link (u,a) is a positive attribute link if a is a positive attribute of node u and it is a negative attribute link otherwise.

Links between mutually exclusive attribute nodes are called mutex links.

The fact that no social node can be linked to multiple mutually exclusive attribute is encoded in the mutex property that is there is no triangle consisting of a mutex link and two positive attribute links in any social attribute network.

Illustration of a social attribute network (SAN)



Graph Theory Application In Recommendation System

Recommender systems have become one of the important tools in e-commerce. They combine the rating of services or product by one user with the rating from other users to answer similar interest queries with predictions and suggestions. Recommender system helps customers by suggesting a probable list of products from which they can easily select the right one. They make customers aware of new and similar products available for purchase by providing comparable cost features, delivery items etc. for example Amazon suggest products based on rating of that product.

Recommender systems have become an integral part of e-commerce sites and other businesses like social networking, movie/music rendering sites, for example Netflix suggest movies, Facebook suggest friends based on similarities and interest, Google Scholar recommends a relevant article based on the area of expertise of researchers, and music applications such as iTunes, YouTube and Spotify suggest next song that the user may like to hear based on the current song a user is listening to.

They have a huge impact on the revenue earned by these businesses and also benefit users by reducing the cognitive load of searching and shifting through an overload of data. Here, we tried to draw a graph through the various items rated by the users and finds the items that are most common among the user and his friends which is then recommended to him. To explain this properly, let us take an example, let us consider the different genres of music. These include rock, Jazz, metal, Death metal, alternatives, Blues, classical etc. Now let us assume that a certain person Ram likes rock, metal and jazz. His friend Sarah likes Rock, death metal and classical while his other friend Ramesh like Rock, metal and alternative. We use our technique to see the common genres of music between Ram and his friends. We see that both Sarah and Ramesh like rock along with Ram. However Ramesh also like metal which is in common with Ram's taste but not like with Sarah who prefer death metal. So, we select Ramesh as the person whose genres will be recommended to Ram

We see that Ramesh also likes alternative since both Ram and Ramesh have two genres in common, there is a high chance that Ram like alternative as well but is probably not aware of it. So, the genre alternative is recommended to ram. Here, it is worth noting that Sarah's choice of death metal and classical could also be recommended but since Ramesh has two genres in common, there is a higher probability of Ram liking alternative then death metal and classical.

Table: Classification of users according to different music genre

Music / Genres	RAM	RAMESH	SARAH
ALTERNATIVE	N	Y	N
ROCK	Y	Y	Y
METAL	Y	Y	N
JAZZ	Y	N	N
DEATH METAL	N	N	Y
CLASSICAL	N	N	Y

In the table N represents does not like genres and Y represents like genres of music.

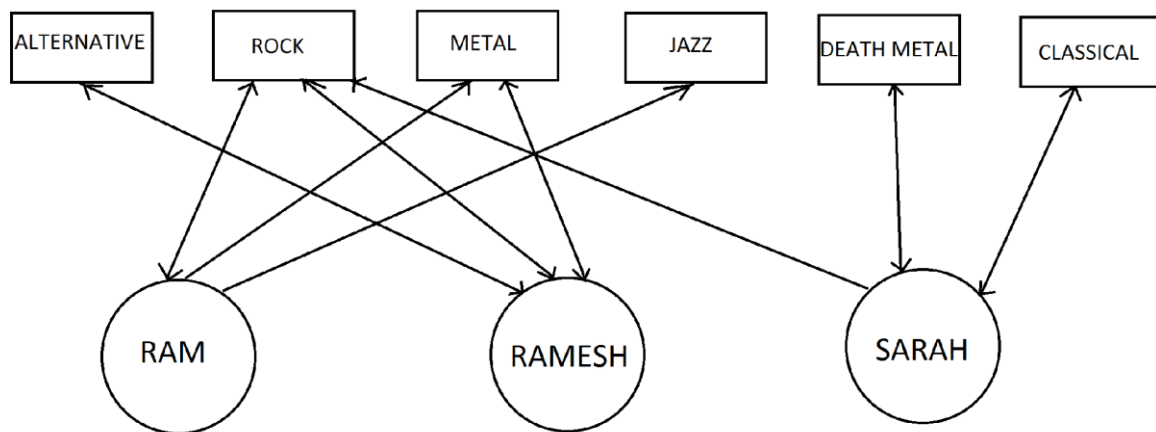


Figure: Example of bipartite graph between customers and music genres

8.1. Recommender systems can be built with two different methods: Content based filtering and collaborative filtering.

8.2. CONTENT BASED FILTERING:

In content based filtering, the similarity between different products is calculated on the basis of the attributes of the products. For instance, in a content based book recommender system, the similarity between the books is calculated on the basis of genres, the author of the book, the publisher of the book, title of the book etc.

8.3. Collaborative Filtering:

Collaborative filtering models which are based on assumption that people like things similar to other things they like, and things that are liked by the other people with similar taste.

Collaborative filtering models are of two types

- I. Nearest Neighbour
- II. Matrix Factorization

8.3a. Nearest Neighbour Collaborative Filtering:- In this type of recommendation system are recommended based on nearest neighbours, nearest neighbour approach is used to find out either similar users or similar products, It can be looked at two ways:-

1. User Based Filtering

2. Item Based Filtering

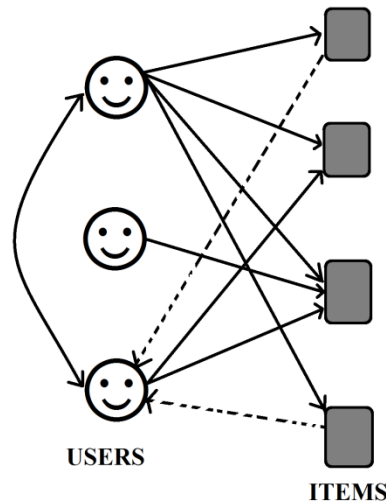


FIG: 1. USER BASED FILTERING

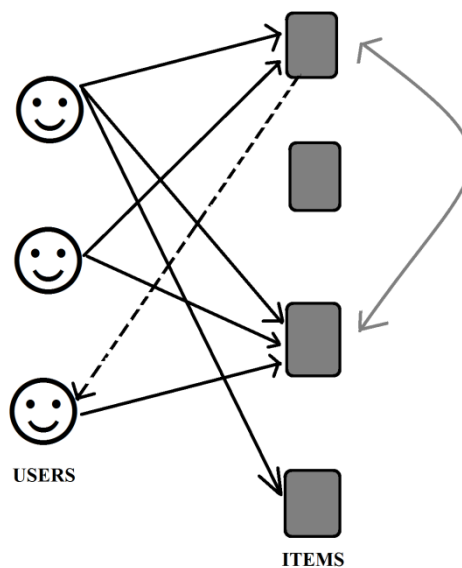


FIG: 2. ITEM BASED FILTERING

8.3a(1).User Based Collaborative Filtering:

Find the users who have similar taste of product as the current user, similarity is based on purchasing behavior of the users; so based on the neighbour purchasing behavior we can recommend items to the current user.

8.3a(2) Item Based Collaborative Filtering:

Recommended items that are similar to the item user bought, similarity is based on co-occurrence of purchases. Item A and B were purchased by both user X and Y then both are similar.

8.3b. Matrix Factorization:

This is basically model based collaborative filtering and matrix factorization is the important technique in a recommendation system.

Let us take an abstractive explanation for the matrix factorization,

When a user gives feedback to a certain movie they saw (they can rate from 1 to 5), this collection of feedback can be represented in a form of a matrix. Where each row represent each user, while each column represents different movies.

Obviously the matrix will be sparse since not everyone is going to watch every movie (we all have different taste when it comes to movies).

GENRE OF MOVIE	M1	M2	M3	M4	M5
COMEDY	3	1	1	3	1
ACTION	1	2	4	1	3

USERS	COMEDY	ACTION
A ☺	√	X
B ☺	X	√
C ☺	√	X
D ☺	√	√

MOVIES/USERS	M1	M2	M3	M4	M5
A ☺	3	1	1	3	1
B ☺	1	2	4	1	3
C ☺	3	1	1	3	1
D ☺	4	3	5	4	4

Fig: Matrix Factorization

9. DISCUSSION

Social Network Analysis (SNA) provides a powerful framework for understanding relationships, interactions, and information flows within social networks. Using graph theory, individuals and organizations can be represented as nodes connected by edges that describe relationships such as friendship, communication, or collaboration. Social graphs and labelled graphs help represent complex relationships and associated information. Community clustering identifies groups with common interests or behaviours, while information diffusion analysis examines how information spreads through networks. Measures such as degree centrality identify influential nodes, and modularity evaluates community structures. These techniques have applications in business, recommendation systems, network optimization, and law enforcement. Thus, SNA offers valuable mathematical and visual tools for analysing modern social networks.

10. CONCLUSION

The main aim of this paper is to present the importance of graph theoretical ideas in various areas of computer applications for researchers that they can use graph theoretical concepts for the research. An overview is presented especially to project the idea of graph theory. So, the graph theory section of each paper is given more importance than to the other sections. Researchers may get some information related to graph theory and its applications in social networks and can get some ideas related to their field of research.

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